JOURNAL OF APPLIED POLYMER SCIENCE VOL. 21, 1459–1467 (1977)

Load–Deflection Relations of Long Cylindrical Rubber Bush Mountings Constructed from Rectangular Blocks

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Synopsis

Theoretical load–deflection relations are given for the four principal modes of deflection of bonded circular rubber bush mountings which are constructed from rectangular blocks of rubber and are sufficiently long so that end effects can be ignored. These relations are applicable to the incompressible neo-Hookean and Mooney materials and are based upon a well-known, exact solution for the bending and stretching of a rectangular block into a circular cylindrical tube. Typical numerical values are tabulated and illustrated graphically.

INTRODUCTION

For unstressed circular cylindrical rubber bush mountings, Adkins and Gent[1] considered four principal modes of deflection, which they termed torsional, axial, radial, and tilting. In a recent paper, Hill[2] considered the effect of precompression on these modes of deflection. In this paper, we derive the corresponding load–deflection relations for bonded circular rubber bushes which are formed by bending and stretching a rectangular rubber block into a cylindrical tube. We suppose that the ends of the block are bonded along the "join" while the inner and outer curved surfaces of the tube are bonded in the usual way to rigid metal cylinders. The four principal modes of deflection are produced by fixing the outer metal cylinder while the inner one undergoes the following displacements: (i) a rotation about its axis (torsional), (ii) a translation in which each point moves parallel to the axis (axial), (iii) a translation in which each point moves through an equal distance in a radial direction (radial), and (iv) a rotation of the axis in a radial plane about a point on itself midway between the plane ends of the tube (tilting or conical).

For homogeneous isotropic incompressible hyperelastic materials, the bending and stretching of a rectangular block into a circular cylindrical tube is described by a well-known, exact solution due to Rivlin.[3] If we assume the bush to be sufficiently long so that end effects can be ignored, then relatively simple load–deflection relations can be derived for the four principal modes of deflection which are superimposed upon the initial deformation. These relations, which have not been given previously, are derived briefly in the Appendix for the neo-Hookean and Mooney materials.

In the following section we define the geometry of the bush, and we note the

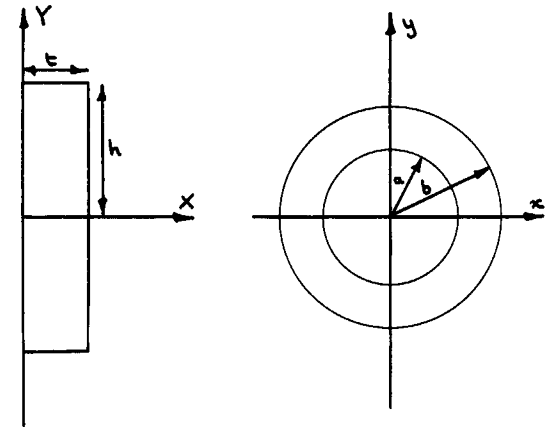

Fig. 1. Bending and stretching of a rectangular block into a circular cylindrical tube.

deformation which describes the bending and stretching of a rubber block into a circular cylindrical tube. In the section thereafter, we summarize the load–deflection relations for the four principal deformation modes which are superimposed upon (1). Also in this section, we note the approximate load–deflection relations which are applicable to extremely thin sheets of rubber. These approximate relations can be identified with the corresponding approximate relations for "thin" unstressed rubber bushes which can be deduced from results given in either Adkins and Gent[1] or Hill.[2] In the final section, we illustrate the load–deflection relations with typical numerical values.

THEORETICAL PRELIMINARIES

We suppose that an undeformed rectangular rubber block of length L, height $2h$, and thickness t is deformed as indicated in Figure 1 into a circular cylindrical tube of inner and outer radii a and b, respectively, by the deformation

$$r = \left[b^2 - \frac{(b^2 - a^2)}{t} X \right]^{1/2}, \qquad \theta = \frac{\pi}{h}(h - Y), \qquad z = \lambda Z \tag{1}$$

where (X,Y,Z) are rectangular Cartesian coordinates in the undeformed body, (r,θ,z) are cylindrical polar coordinates in the deformed body, and λ is a constant given by

$$\lambda = \frac{2th}{\pi(b^2 - a^2)} \tag{2}$$

We note that the new length of the tube is λL and that we shall assume the initial and final cross sections are given so that λ is determined by eq. (2). We now suppose that the tube is simultaneously bonded along the "join" and along its curved surfaces to rigid metal cylinders. Load–deflection relations for the four principal modes of deflection are derived in the Appendix and summarized in the following section. We remark that Rivlin[3] has shown that eq. (1) is valid for any homogeneous isotropic incompressible hyperelastic material. However, we shall only consider the case of the neo-Hookean and Mooney materials, which are standard prototypes for rubber-like materials. The Mooney form of the strain–energy function is given by

$$\sum = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{3}$$

where I_1 and I_2 are the principal invariants of the finite deformation strain tensor, and C_1 and C_2 are material constants for which $C_2 = 0$ for the neo-Hookean material.

LOAD-DEFLECTION RELATIONS

Before summarizing the load–deflection relations which are derived in the Appendix, we make the following points. Firstly, we remind the reader that eqs. (4), (5), (6), and (7) are "exact" results for bushes which are sufficiently long so that end effects can be ignored, whereas the load–deflection relation (10) for tilting deformations is only an "estimate" based on an approximate argument given originally by Adkins and Gent.[1] Secondly, we remark that the results for torsional and axial deflections are valid for large strains, whereas the radial and tilting relations are applicable only for small deflections. Finally, we mention that we have added an asterisk to the forces and moments given below so as to avoid any confusion with the corresponding results given in Hill.[2]

Torsional Deflections

The couple M^* required to rotate the inner metal cylinder about its axis through an angle θ_0 which is not necessarily small is given by

$$M^* = 2(C_1 + \lambda^2 C_2)\frac{h(b^2 - a^2)L\theta_0}{t \log (b/a)} \tag{4}$$

Axial Deflections

For the Mooney material, the force F^* required to displace the inner metal cylinder through a distance z_0 parallel to its axis is given by

$$F^* = \frac{8\pi C_2 L z_0}{\lambda \log \left(\dfrac{C_1 h^2 + C_2\pi^2 b^2}{C_1 h^2 + C_2 \pi^2 a^2}\right)} \qquad (C_2 \neq 0) \tag{5}$$

while for the neo-Hookean material, we have

$$F^* = 4C_1\frac{hLz_0}{t} \qquad (C_2 = 0) \tag{6}$$

and we note that eq. (6) can be derived from (5) if we take the appropriate limit of (5) as C_2 tends to zero.

Radial Deflections

For small radial deformations, the force G^* which is required to displace the inner metal cylinder a distance ϵ uniformly along its length in "any" radial direction is given by

$$G^* = \frac{8\pi(C_1 + \lambda^2 C_2)\phi(\alpha,\beta)L\epsilon}{\int_\alpha^\beta \phi(\alpha,\eta)d\eta \int_\alpha^\beta \phi(\beta,\eta)d\eta - \phi(\alpha,\beta)\int_\alpha^\beta \int_\alpha^\xi \phi(\xi,\eta)d\eta\, d\xi} \tag{7}$$

where α and β are defined by

$$\alpha = \frac{\pi t a^2}{h(b^2 - a^2)}, \qquad \beta = \frac{\pi t b^2}{h(b^2 - a^2)} \tag{8}$$

and the function $\phi(\xi,\eta)$ is given by

$$\phi(\xi,\eta) = \frac{I_1(\xi)K_1(\eta) - I_1(\eta)K_1(\xi)}{(\xi\eta)^{1/2}} \tag{9}$$

where I_1 and K_1 are the usual modified Bessel functions of order one.

Tilting or Conical Deflections

Employing the argument given by Adkins and Gent,[1] we can show that for long bushes, the couple N^* required to rotate the axis of the inner metal cylinder through a small angle δ about its midpoint is given "approximately" by

$$N^* = \frac{1}{12}\left(\frac{G^*}{\epsilon}\right)(\lambda L)^2\,\delta \tag{10}$$

where G^*/ϵ is obtained from eq. (7).

If for thin sheets of rubber for which t/h is small we take λ to be of order unity, then from the above we obtain the following approximate load–deflection relations:

$$M^* \sim 2\mu_0 \frac{ha^2}{t} L\theta_0, \qquad F^* \sim 4(C_1 h^2 + C_2\pi^2 a^2)\frac{Lz_0}{th}$$

$$G^* \sim 12\mu_0 \frac{\pi^2 a^4}{t^3 h} L\epsilon, \qquad N^* \sim \mu_0 \frac{\pi^2 a^4}{t^3 h} L^3\delta \tag{11}$$

where $\mu_0 = 2(C_1 + C_2)$ is the usual linear shear modulus. It is interesting to note that if now we suppose

$$t \sim (b - a), \qquad h \sim \pi a \tag{12}$$

then the relations obtained from eq. (11) agree with those for "thin" unstressed bushes which can be deduced from results given in either Adkins and Gent[1] or Hill.[2]

In the following section we illustrate the above relations by means of the ratios

$$m^* = \frac{M^*}{M_0}, \qquad f^* = \frac{F^*}{F_0}, \qquad g^* = \frac{G^*}{G_0}, \qquad n^* = \frac{N^*}{N_0} \tag{13}$$

where M_0, F_0, G_0, and N_0 are the forces and moments for an unstressed rubber bush of length L and inner and outer radii a and b, respectively. Expressions for these quantities are given by Hill,[2] and we note that these ratios are of order unity if t/h and $(b - a)/h$ are small compared with unity and are of the same magnitude.

TABLE I
Numerical Values of m^*, f^*, g^*, and n^* for Various Values of t/h for the Neo-Hookean Material ($C_2 = 0$) with $b/a = 2.0$ and $h/a = \pi$

Parameter	Numerical values									
t/h	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
m^*	5.17	2.58	1.72	1.29	1.03	0.86	0.74	0.65	0.57	0.52
f^*	4.41	2.21	1.47	1.10	0.88	0.74	0.63	0.55	0.49	0.44
g^*	4.63	2.33	1.57	1.19	0.97	0.83	0.73	0.66	0.60	0.56
n^*	0.05	0.10	0.16	0.21	0.27	0.33	0.39	0.46	0.53	0.61

TABLE II
Numerical Values of m^*, f^*, g^*, and n^* for Various Values of t/h for the Mooney Material ($C_2/C_1 = 0.1$) with $b/a = 2.0$ and $h/a = \pi$

Parameter	Numerical values									
t/h	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
m^*	4.70	2.36	1.58	1.20	0.97	0.81	0.71	0.63	0.57	0.52
f^*	4.99	2.50	1.66	1.25	1.00	0.83	0.71	0.62	0.55	0.50
g^*	4.21	2.13	1.44	1.10	0.91	0.78	0.70	0.64	0.59	0.56
n^*	0.05	0.09	0.14	0.19	0.25	0.31	0.38	0.45	0.53	0.62

NUMERICAL RESULTS

In order to illustrate the above load–deflection relations, we consider the ratios b/a and h/a to be fixed, and we vary t/h. With $b/a = 2.0$ and $h/a = \pi$, typical numerical values of m^*, f^*, g^*, and n^* are given in Tables I and II for the neo-Hookean ($C_2 = 0$) and Mooney ($C_2/C_1 = 0.1$) materials, respectively. These results are illustrated graphically in Figures 2 and 3. From these results, we see that m^*, f^*, and g^* vary uniformly with t/h and, moreover, are significantly greater than unity for small values of t/h. However, these increases appear to be at the expense of a significant decrease in n^*. These low values of n^* are due primarily to λ being small, since from eqs. (10) and (13) we have

$$n^* = \lambda^2 g^* \tag{14}$$

and from eq. (2) we have, for the particular ratios $b/a = 2.0$ and $h/a = \pi$,

$$\lambda = \frac{2\pi}{3}\left(\frac{t}{h}\right) \tag{15}$$

Thus for small t/h, λ is small and hence, from eq. (14), n^* is small. We again remind the reader that (10) is only an "estimate" for N^* and that considering the uniform behavior of the three ratios m^*, f^*, and g^*, the result (10) may well be unreliable. Certainly, the feature that n^* is decreased whenever m^*, f^*, and g^* are increased is a typical characteristic of the load–deflection relations given above. This decrease in n^* may well be a practical characteristic of such bushes, although it may not be as severe as that suggested from the above results.

In conclusion, for bonded cylindrical rubber bush mountings which are formed by bending and stretching a rectangular block of rubber into a cylindrical tube,

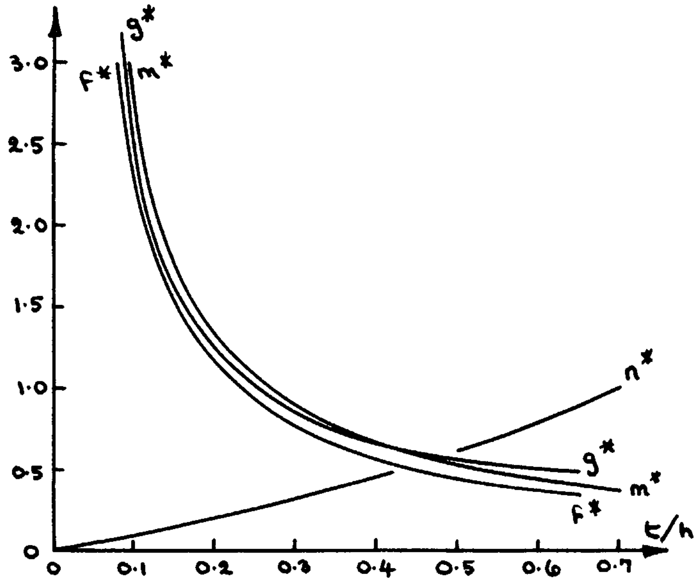

Fig. 2. Variation of m^*, f^*, g^*, and n^* with t/h for the neo-Hookean material ($C_2 = 0$) with $b/a = 2.0$ and $h/a = \pi$.

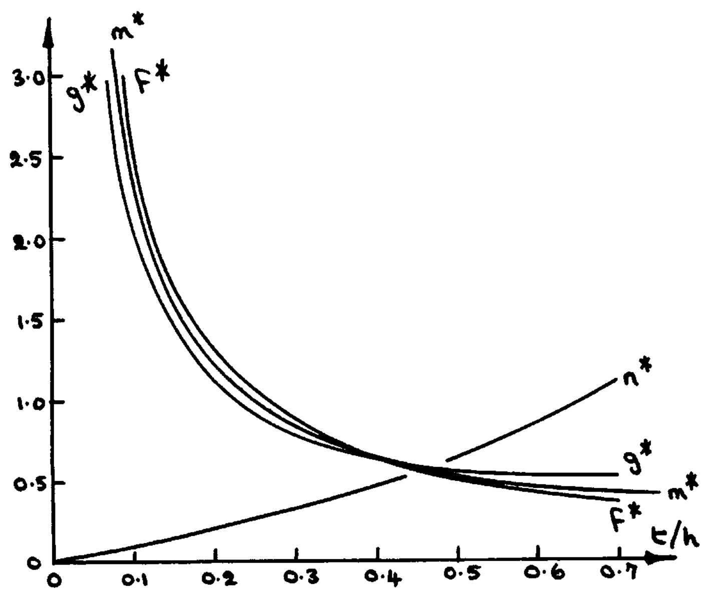

Fig. 3. Variation of m^*, f^*, g^*, and n^* with t/h for the Mooney material ($C_2/C_1 = 0.1$) with $b/a = 2.0$ and $h/a = \pi$.

load–deflection relations have been given for the four principal deformation modes. These relations are applicable to long bushes and are valid for the incompressible neo-Hookean and Mooney materials. Numerical results are given which indicate that the stiffnesses for the torsional, axial, and radial modes can be considerably increased using bushes of this type. However, it would appear that these increases are accompanied by a decrease in the tilting stiffness. Whether or not this decrease occurs in practice will have to be decided by experiment.

Appendix

In this Appendix, we give a brief derivation of the load–deflection relations (4) and (7) for torsional and radial deflections. For axial deflections, we indicate how (5) and (6) can be deduced, while for the tilting load–deflection relation (10), we refer the reader to either Adkins and Gent[1] or Hill.[2] Before considering the load–deflection relations, we summarize the basic equations for large elastic plane deformations of a Mooney material. For material rectangular Cartesian coordinates (X,Y,Z) and spatial cylindrical polar coordinates (r,θ,z), we consider the plane deformation

$$r = r(X,Y), \qquad \theta = \theta(X,Y), \qquad z = \lambda Z \tag{A1}$$

where λ is a positive constant. For an isotropic incompressible Mooney material, (A1) satisfies the following:

$$\begin{aligned} r_X\theta_Y - r_Y\theta_X &= \frac{1}{\lambda r} \\ p_r^* &= \mu\{\nabla^2 r - r(\theta_X^2 + \theta_Y^2)\} \\ p_\theta^* &= \mu r^2\left\{\nabla^2\theta + \frac{2}{r}(r_X\theta_X + r_Y\theta_Y)\right\} \end{aligned} \tag{A2}$$

where subscripts denote partial differentiation, p^* is the pressure function associated with incompressible materials, $\mu = 2(C_1 + \lambda^2 C_2)$, and ∇^2 is the two-dimensional Laplacian given by

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}. \tag{A3}$$

For a derivation of these equations and for the associated stress components t^{ij}, the reader is referred to Hill.[4] Using these stress components, we can show that the force G^*, which must be applied in the direction $\theta = 0$ to a curve which was originally given by the straight line X = constant, is given by

$$G^* = L\int_{-h}^{h}[-\lambda p(r\sin\theta)_Y + \mu(r\cos\theta)_X]dY \tag{A4}$$

where the lengths L and h are those defined previously.

If we write the deformation (1) as

$$r = (AX + B)^{1/2}, \qquad \theta = CY + D, \qquad z = \lambda Z \tag{A5}$$

where A, B, C, and D are constants which are readily identified from (1), then we can show that (A5) is a solution of (A2) where the pressure function is given by

$$p_0(X) = \frac{\mu A}{2}\left\{\frac{A}{4(AX+B)} - C^2 X\right\} + \sigma \tag{A6}$$

where σ is a constant. If $t_0{}^{ij}$ denotes the stress tensor for (A5), then we assume that the constant σ is determined by the condition

$$\int_a^b t_0^{33}\, r dr = 0 \tag{A7}$$

which approximates the boundary condition of zero surface tractions on the plane ends of the cylindrical rubber tube.

For torsional deflections superimposed upon (A5), we look for a solution of (A2) of the form

$$r = (AX + B)^{1/2}, \qquad \theta = CY + D + f(X), \qquad z = \lambda Z \tag{A8}$$

where $f(X)$ is a function of X only which satisfies the displacement boundary conditions

$$f(0) = 0, \qquad f(t) = \theta_0. \tag{A9}$$

From (A2) and (A8), we find that

$$f(X) = \alpha_1 \log (AX + B) + \alpha_2 \tag{A10}$$

where α_1 and α_2 are constants which are determined by (A9). On evaluating these constants and the stress tensor associated with (A8), it is a simple matter to show that the required couple M^* is given by (4).

In a similar manner for axial deflections we look for a solution of the three-dimensional equations of finite elasticity of the form

$$r = (AX + B)^{1/2}, \qquad \theta = CY + D, \qquad z = \lambda Z + g(X) \tag{A11}$$

where $g(X)$ is a function of X only which satisfies the displacement boundary conditions

$$g(0) = 0, \qquad g(t) = z_0 \tag{A12}$$

We find that for the Mooney material we have

$$g(X) = \beta_1 \log [C_1 + C^2C_2(AX + B)] + \beta_2 \qquad (C_2 \neq 0) \tag{A13}$$

while for the neo-Hookean material we obtain

$$g(X) = \gamma_1 X + \gamma_2 \qquad (C_2 = 0) \tag{A14}$$

where β_1, β_2, γ_1, and γ_2 are constants which are determined by (A12). Omitting the details, (A13) and (A14) can be shown to give rise to the load–deflection relations (5) and (6), respectively. We note that the relations (4), (5), and (6) have not been given previously, although the deformations of the form (A8) and (A11) have been considered by previous authors. For further references and for the general theory leading to (A13) and (A14), the reader is referred to Truesdell and Noll, page 201.[5]

For radial deformations, we suppose ϵ is the small distance moved by the inner cylinder in the direction $\theta = 0$, and we look for solutions of (A2) of the form

$$\begin{aligned} r &= (AX + B)^{1/2} + \epsilon u(x) \cos (CY + D) \\ \theta &= CY + D + \epsilon v(x) k^{1/2} \sin (CY + D) \\ p^* &= p_0(X) + \epsilon p(x) A^2 k^{3/2} \cos (CY + D) \end{aligned} \tag{A15}$$

where $k = C/A$ and u, v, and p are functions of x only which is defined by

$$x = k(AX + B). \tag{A16}$$

In terms of these functions, the displacement boundary conditions at the inner and outer cylinders can be shown to become

$$\begin{aligned} u(\alpha) &= 1, \qquad v(\alpha) = -\alpha^{-1/2} \\ u(\beta) &= 0, \qquad v(\beta) = 0 \end{aligned} \tag{A17}$$

where α and β are defined by (8). We note here that, by considering a radial displacement in an arbitrary radial direction, we can show that the load–deflection relation (7) is independent of this arbitrary direction, and therefore there is no loss of generality in considering displacements in the direction $\theta = 0$.

On substituting (A15) into (A2) and (A4) and considering only terms of order ϵ, we obtain the following:

$$\begin{aligned} x^{1/2}u' + \frac{u}{2x^{1/2}} + \frac{v}{2} &= 0 \\ 2x^{1/2}p' &= \mu \left\{ u'' + \left(2x - \frac{1}{2x} \right) u' \right\} \\ -p &= \mu x \left\{ v'' + \frac{v'}{x} + \frac{u}{4x^{5/2}} + 2x^{1/2}u' \right\} \\ G^* &= -\pi \epsilon L[-2x^{1/2}p + \mu(u - x^{1/2}v)'] \end{aligned} \tag{A18}$$

where primes denote differentiation with respect to x. The force G^* must be independent of x, and thus we obtain immediately the first integral

$$-2x^{1/2}p + \mu(u - x^{1/2}v)' = -4\mu\delta_1 \tag{A19}$$

where δ_1 is a constant and $(A18)_4$ becomes

$$G^* = 4\pi\mu\epsilon L\delta_1 \tag{A20}$$

If we now eliminate v from (A19) by means of $(A18)_1$ and then substitute the resulting expression for p in $(A18)_2$, we obtain after some simplifications

$$x^2w'' + xw' - (1 + x^2)w = \delta_1 x^{1/2} \tag{A21}$$

where $w = x^{1/2}u'$. The solution for u can be shown to be given by

$$u(x) = \delta_1 \int_\alpha^x \int_\alpha^\xi \frac{I_1(\xi)K_1(\eta) - I_1(\eta)K_1(\xi)}{(\xi\eta)^{1/2}} d\eta\, d\xi + \delta_2 \int_\alpha^x \frac{I_1(\xi)}{\xi^{1/2}} d\xi + \delta_3 \int_\alpha^x \frac{K_1(\xi)}{\xi^{1/2}} d\xi + \delta_4 \tag{A22}$$

where δ_2, δ_3, and δ_4 are further arbitrary constants and I_1 and K_1 are the usual modified Bessel functions of order one.

If now from $(A18)_1$ and (A22) we obtain an expression for $v(x)$, then from this result and (A22) we can show from the boundary conditions (A17) that δ_1 is given by

$$\delta_1 = \frac{\phi(\alpha,\beta)}{\left\{\int_\alpha^\beta \phi(\alpha,\eta)d\eta \int_\alpha^\beta \phi(\beta,\eta)d\eta - \phi(\alpha,\beta) \int_\alpha^\beta \int_\alpha^\xi \phi(\xi,\eta)d\eta\, d\xi\right\}} \tag{A23}$$

where $\phi(\xi,\eta)$ is defined by (9). Thus, from (A20) and (A23), we obtain the required load–deflection relation (7) for small radial deformations.

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Received December 10, 1975
Revised April 12, 1976